Oscillation of Nonlinear Neutral Type Second Order Delay Difference Equations

E. Thandapani and M.Vijaya

Abstract--- *In this paper we present some new oscillation criteria for second order nonlinear neutral type delay difference equation of the form*

$$\Delta(a_n\,(\Delta(x_n + c_n\,x_{n-k}))^{\alpha}) + q_n\,x_{n+1-\ell}^{\beta} = 0, n \ge n_0 \ge 0,$$

Where, Δ *is a forward difference operator defined by* $\Delta x_n = x_{n+1} - x_n$, $n \in N(n_0) = (n_0, n_0 + 1, \dots)$, n_0 *a nonnegative integer,* k *and* ℓ *are positive integers,* α *and* β *are the ratio of odd positive integers,* $\{a_n\}, \{c_n\}$ *and* $\{q_n\}$ *are real sequences. Examples are provided to illustrate the results.*

Keywords--- *Nonlinear, Neutral Delay Difference Equation, Oscillation, Second Order*

I. Introduction

CONSIDER a second order nonlinear neutral delay difference equation of the form

$$\Delta(a_n\,(\Delta(x_n + c_n\,x_{n-k}))^{\alpha}) + q_n\,x_{n+1-\ell}^{\beta} = 0, \qquad (1)$$

Where $n \in N(n_0) = (n_0, n_0 + 1, \dots)$, n_0 a nonnegative integer, ℓ and k are positive integers, α and β are ratio of odd positive integers, $\{a_n\}, \{c_n\}$ and $\{q_n\}$ are real sequences defined for $n \in N(n_0)$.

Let $\theta = \max\{k, \ell\}$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$ and satisfies equation (1) for all $n \in N(n_0)$. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In the last few decades, there has been an increasing interest in establishing sufficient conditions for the oscillation and nonoscillation of solutions of different classes of second order neutral type delay difference equations, see for example [1, 2, 4,12], and the references cited therein.

In [3, 5, 6, 7, 8, 9, 10,11], the authors considered the equation of the type (1) and established sufficient conditions for the oscillation of all solutions of equation (1) using Riccati type transformation and Integral averaging method. Further, they have established some sufficient conditions for the oscillation of all solutions of equation (1) using Philos type oscillation criteria.

In this paper, we use the generalized Riccati type transformation and Schwarz inequality to establish some new sufficient conditions for the oscillation of all solutions of equation (1). In Section 2, we present sufficient conditions for the oscillation of all solutions of equation (1) when α and β satisfy different set of conditions. In Section 3, we provide examples to illustrate the main results.

II. Oscillation Results

In this section, we establish some new criteria for the oscillation of all solutions of equation (1). We assume that

(c_1) $0 \le c_n < 1$;

(c_2) $\{q_n\}$ is a nonnegative real sequence and $\{a_n\}$ is a positive real sequence with

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty \quad .$$

We begin with the following theorem.

Theorem 2.1: Let $\alpha = 1$ and $\beta > 1$. Suppose there exist a positive sequence $\{u_n\}$ and a nonnegative sequence $\{\phi_n\}$ such that

$$\Delta(a_{n-1-\ell}\,\Delta u_n) \le 0 \quad for \;\; n \in N(n_0), \qquad (2)$$

and

$$\sum_{n=n_0}^{\infty}\left[\psi_n - \frac{u_n}{K_n}\left(\phi_{n+1} + \frac{K_n\,\Delta u_n}{2u_n}\right)^2\right] = \infty,\; n \in N(n_0), \quad (3)$$

where

$$K_n = M_1^2\,a_{n-\ell}\sum_{s=n_0}^{n-\ell}\frac{a_{s-\ell}}{a_s^2\,u_s},\; Q_n = q_n\,(1 - c_{n+1-\ell})^{\beta},$$

$$\psi_n = u_n\left(Q_n + \frac{(\phi_{n+1})^2}{K_n} - \Delta\phi_n\right)$$

for some positive constant M_1 and $n \in N(n_0)$, then every solution of equation (1) is oscillatory.

Proof: Assume to the contrary that equation (1) possesses a nonoscillatory solution $\{x_n\}$ for all $n \in N(n_0)$. Without

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loss of generality, we shall assume that $x_{n-\theta} > 0$ for all $n \ge n_1 \ge n_0 + \theta.$

Let $z_n = x_n + c_n \, x_{n-k}.$ (4)

Then in view of condition (c_1), we have $z_n > 0, \Delta(a_n\,(\Delta z_n)) \le 0$ for all $n \ge n_1$. Therefore $\{a_n(\Delta z_n)\}$ is a decreasing sequence, and in view of condition (c_2), we have $\Delta z_n > 0$ for $n \ge n_1$. Hence

$$z_n > 0, \;\; \Delta z_n > 0, \;\; \Delta(a_n\,(\Delta z_n)) \le 0, \quad for \;\; n \ge n_1. \quad (5)$$

From (4) and (5), we obtain

$$x_n = z_n - c_n\, x_{n-k} \;\ge\; z_n - c_n\, z_{n-k} \;\ge\; (1-c_n)\, z_n. \qquad (6)$$

Using (6) in equation (1), and using the definition of Q_n, we have

$$\Delta(a_n\,(\Delta z_n)) + Q_n\, z_{n+1-\ell}^{\beta} \le 0, \;\; n \ge n_1. \qquad (7)$$

Define

$$w_n = u_n \left[\frac{a_n\,\Delta z_n}{z_{n-\ell}^{\beta}} + \phi_n \right] \qquad (8)$$

Then $w_n > 0$ and

$$\Delta w_n \le \frac{\Delta u_n}{u_{n+1}}\, w_{n+1} + u_n\left(-Q_n - \frac{a_n\,\Delta z_n\,\Delta z_{n-\ell}^{\beta}}{z_{n+1-\ell}^{\beta}\, z_{n-\ell}^{\beta}} + \Delta\phi_n \right). \quad (9)$$

By Mean Value Theorem[1], we have

$$\Delta z_{n-\ell}^{\beta} \ge \beta\, z_{n-\ell}^{\beta-1}\, \Delta z_{n-\ell},$$

and using the last inequality in (9), we obtain

$$\Delta w_n \le \frac{\Delta u_n}{u_{n+1}}\, w_{n+1} + u_n\left(-Q_n + \Delta\phi_n - \frac{\beta}{a_{n-\ell}}\left(\frac{a_{n+1}\,(\Delta z_{n+1})}{z_{n+1-\ell}^{\gamma}}\right)^2 \right), \qquad (10)$$

where $\gamma = \dfrac{\beta+1}{2}$. From (10), we have

$$\Delta\left(\frac{u_n\, a_n\, \Delta z_n}{z_{n-\ell}^{\beta}}\right) \le -u_n Q_n + \frac{\Delta u_n\, a_{n+1}\, \Delta z_{n+1}}{z_{n+1-\ell}^{\beta}} - \frac{\beta\, u_n}{a_{n-\ell}}\left(\frac{a_{n+1}\,\Delta z_{n+1}}{z_{n+1-\ell}^{\gamma}}\right)^2.$$

Summing the last inequality from n_1 to $n-1$ and using condition (2), we obtain

$$\frac{u_n a_n \Delta z_n}{z_{n-\ell}^{\beta}} \le \frac{u_{n_1} a_{n_1} \Delta z_{n_1}}{z_{n_1-\ell}^{\beta}} + \Delta u_{n_1} a_{n_1-\ell} \sum_{s=n_1}^{n-1} \frac{\Delta z_{s-\ell}}{z_{s+1-\ell}^{\beta}}$$

$$- \sum_{s=n_1}^{n-1} u_s\, Q_s - \sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_{s+1}\,\Delta z_{s+1}}{z_{s+1-\ell}^{\gamma}}\right)^2. \qquad (11)$$

Since $z_{n-\ell} \le t \le z_{n+1-\ell}$, we have

$$\frac{\Delta z_{n-\ell}}{z_{n+1-\ell}^{\beta}} \le \int_{z_{n-\ell}}^{z_{n+1-\ell}} \frac{dt}{t^{\beta}}.$$

Using the above inequality in (11) and then summing, we obtain

$$\frac{u_n\, a_n\, \Delta z_n}{z_{n-\ell}^{\beta}} \le L - \sum_{s=n_1}^{n-1} u_s\, Q_s - \sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_{s+1}\,\Delta z_{s+1}}{z_{s+1-\ell}^{\gamma}}\right)^2, \quad (12)$$

where $L = \dfrac{u_{n_1}\, a_{n_1}\, \Delta z_{n_1}}{z_{n_1+1-\ell}^{\beta}} + \dfrac{\Delta u_{n_1} a_{n_1-\ell}}{\beta-1}\, z_{n_1-\ell}^{1-\beta}.$

Since $\dfrac{u_n\, a_n\, \Delta z_n}{z_{n-\ell}^{\beta}} > 0$, we have from (12)

$$\sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_{s+1}\,\Delta z_{s+1}}{z_{s+1-\ell}^{\gamma}}\right)^2 \; converges\, as \;\; n \to \infty.$$

From (5), the sum

$$\sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_s\,\Delta z_s}{z_s^{\gamma}}\right)^2 \; is\; also\; converges\, as \;\; n \to \infty,$$

and hence there exists a constant M, such that

$$\sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_s\,\Delta z_s}{z_s^{\gamma}}\right)^2 \le M \;\; for\; all \;\; n \ge n_1. \qquad (13)$$

By Schwarz's inequality, we obtain using (13)

$$\left| \int_{z_{n_1}}^{z_n} \frac{dt}{t^{r}} \right|^2 \le \left| \sum_{s=n_1}^{n-1} \frac{\Delta z_s}{z_s^{\gamma}} \right|^2$$

$$= \left| \sum_{s=n_1}^{n-1} \sqrt{\frac{a_{s-\ell}}{\beta\, u_s a_s^2}}\, \sqrt{\frac{\beta\, u_s}{a_{s-\ell}}}\, \frac{a_s \Delta z_s}{z_s^{\gamma}} \right|^2$$

$$\le \sum_{s=n_1}^{n-1}\left(\frac{a_{s-\ell}}{\beta\, u_s\, a_s^2}\right) \sum_{s=n_1}^{n-1} \frac{\beta\, u_s}{a_{s-\ell}}\left(\frac{a_s\,\Delta z_s}{z_s^{\gamma}}\right)^2$$

$$\le M \sum_{s=n_1}^{n-1} \frac{a_{s-\ell}}{\beta\, u_s\, a_s^2}.$$

Hence, for $n \ge n_1$, we have

$$\left|z_n^{1-\gamma} - z_{n_1}^{1-\gamma}\right| \le (1-\gamma)\, M^{\frac{1}{2}} \left(\sum_{s=n_1}^{n-1} \frac{a_{s-\ell}}{\beta\, u_s\, a_s^2} \right)^{\frac{1}{2}}.$$

Therefore, there exist a constant M_1, and an integer $n_2 > n_1$ such that

$$\left|z_n^{1-\gamma}\right| \le M_1 \left(\sum_{s=n_1}^{n-1} \frac{a_{s-\ell}}{\beta\, u_s\, a_s^2} \right)^{\frac{1}{2}} \;\; for\;\; n \ge n_2.$$

Then for all $n \ge N \ge n_2 \ge n_1 + \ell - 1$, we have

$$\left|z_{n+1-\ell}^{1-\gamma}\right| \le M_1 \left(\sum_{s=n_1}^{n-\ell} \frac{a_{s-\ell}}{\beta\, u_s\, a_s^2} \right)^{\frac{1}{2}}$$

or

$$\left|z_{n+1-\ell}^{\gamma}\right| \le \left|z_{n+1-\ell}^{\beta}\right| M_1 \left(\sum_{s=n_1}^{n-\ell} \frac{a_{s-\ell}}{\beta\, u_s\, a_s^2} \right)^{\frac{1}{2}} \;\; for\;\; n \ge N. \qquad (14)$$

From (14) and (10), we have

$$\Delta w_n \le \frac{\Delta u_n}{u_{n+1}}\, w_{n+1} + u_n \left(-Q_n + \Delta\phi_n - \frac{1}{K_n} \left(\frac{a_{n+1}\,\Delta z_{n+1}}{z_{n+1-\ell}^{\beta}} \right)^2 \right)$$

or

$$\Delta w_n \le \frac{\Delta u_n}{u_{n+1}}\, w_{n+1} + u_n \left(-Q_n + \Delta\phi_n - \frac{1}{K_n} \left(\frac{w_{n+1}}{u_{n+1}} - \phi_{n+1} \right)^2 \right)$$

$$= -\psi_n - \frac{u_n}{K_n u_{n+1}^2}\, w_{n+1}^2 + \frac{u_n}{u_{n+1}} \left(\frac{2\phi_{n+1}}{K_n} + \frac{\Delta u_n}{u_n} \right) w_{n+1}$$

$$\le -\psi_n + \frac{u_n}{K_n} \left(\phi_{n+1} + \frac{K_n\, \Delta u_n}{2u_n} \right)^2.$$

Summing the last inequality from N to $n-1$, we obtain

$$\sum_{s=N}^{n-1} \left[\psi_s - \frac{u_s}{K_s} \left(\phi_{s+1} + \frac{K_s \Delta u_s}{2u_s} \right)^2 \right] \le -\sum_{s=N}^{n-1} \Delta w_s$$

$$= -w_n + w_N \le w_N$$

for large values of n. This contradicts the condition (3), and now the proof is complete.

Corollary 1: Let the assumptions of Theorem 2.1 hold. If assumption (3) is replaced by the two conditions

$$\sum_{n=n_0}^{\infty} \psi_n = \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{u_n}{K_n} \left(\phi_{n+1} + \frac{K_n\, \Delta u_n}{2u_n} \right)^2 < \infty,$$

then every solution of equation (1) is oscillatory.

Corollary 2: If we choose $\phi_n = 0$, in Corollary 1 then $\psi_n = u_n\, Q_n$ and the conclusion of Theorem 2.1 remains valid if assumption (3) is replaced by the two conditions

$$\sum_{n=n_0}^{\infty} u_n\, Q_n = \infty, \qquad \text{and} \qquad \sum_{n=n_0}^{\infty} \frac{K_n (\Delta u_n)^2}{4u_n} < \infty.$$

Remark 1: The result obtained in Theorem 2.1 improves Theorem 1 of [3] and Theorem 2.1 of [6].

Theorem 2.:2 Let $\alpha = 1$ and $\beta > 1$. Assume $\Delta(\frac{n}{a_n}) \ge 0$ and $\Delta(a_{n+1-\ell}\, \Delta(\frac{n}{a_n}))\; \le 0$ for all $n \in N(n_0)$. *If*

$$\sum_{n=n_0}^{\infty} \frac{n\, Q_n}{a_n} = \infty, \qquad (16)$$

then every solution of equation (1) is oscillatory.

Proof: Proceeding as in the proof of Theorem 2.1, we arrive at inequality (7), that is,

$$\Delta(a_n\, \Delta z_n) + Q_n\, z_{n+1-\ell}^{\beta} \le 0, \;\; for \;\; n \ge n_1.$$

Multiply the last inequality by $\dfrac{n}{a_n\, z_{n+1-\ell}^{\beta}}$ and then summing from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} \frac{s}{a_s} \frac{\Delta(a_s\, \Delta z_s)}{z_{s+1-\ell}^{\beta}} + \sum_{s=n_1}^{n-1} \frac{sQ_s}{a_s} \le 0.$$

Using summation by parts formula, we obtain

$$\frac{n\Delta z_n}{z_{n+1-\ell}^{\beta}} - \frac{n_1 \Delta z_{n_1}}{z_{n_1+1-\ell}^{\beta}} - \sum_{s=n_1}^{n-1} a_{s+1} \Delta z_{s+1} \Delta\left(\frac{s/a_s}{z_{s+1-\ell}^{\beta}} \right) + \sum_{s=n_1}^{n-1} \frac{s\, Q_s}{a_s} \le 0$$

or

$$-\frac{n_1 \Delta z_{n_1}}{z_{n_1+1-\ell}^{\beta}} - \sum_{s=n_1}^{n-1} \frac{a_{s+1} \Delta z_{s+1}}{z_{s+2-\ell}^{\beta}} \Delta\left(\frac{s}{a_s} \right) + \sum_{s=n_1}^{n-1} \frac{sQ_s}{a_s} \le 0.$$

Then

$$\sum_{s=n_1}^{n-1} \frac{sQ_s}{a_s} \le L + \sum_{s=n_1}^{n-1} \frac{a_{s+1-\ell} \Delta z_{s+1-\ell}}{z_{s+2-\ell}^{\beta}} \Delta\left(\frac{s}{a_s} \right),$$

where $L = \dfrac{n_1 \Delta z_{n_1}}{z_{n_1+1-\ell}^{\beta}}$. Now by hypothesis, we obtain

$$\sum_{s=n_1}^{n-1} \frac{sQ_s}{a_s} \le L + a_{n_1+1-\ell}\; \Delta\left(\frac{n_1}{a_{n_1}} \right) \sum_{s=n_1}^{n-1} \frac{\Delta z_{s+1-\ell}}{z_{s+2-\ell}^{\beta}}. \qquad (17)$$

For $z_{n+1-\ell} \le t \le z_{n+2-\ell}$, we have

$$\int_{z_{n+1-\ell}}^{z_{n+2-\ell}} \frac{dt}{t^{\beta}} \geq \frac{\Delta z_{n+1-\ell}}{z_{n+2-\ell}^{\beta}}. \tag{18}$$

Using (18) in (17) we obtain,

$$\sum_{s=n_1}^{n-1} \frac{sQ_s}{a_s} \leq L + a_{n_1+1-\ell}\, \Delta\left(\frac{n_1}{a_{n_1}}\right) \frac{z_{n_1+1-\ell}^{1-\beta}}{\beta-1} < \infty$$

for large values of n, which contradicts the condition (16). This completes the proof.

Remark 2: Let $a_n = 1$ and $c_n = 0,$ then Theorem 2.2 reduces to Theorem 4.1 of Hooker and Patula [4].

Theorem 2.3: Let $\beta > \alpha$. If

$$\sum_{n=n_0}^{\infty} \left(\frac{Q_n}{a_{n-\ell}}\right)^{\frac{1}{\alpha}} = \infty, \tag{19}$$

then all solutions of equation (1) are oscillatory.

Proof: As in Theorem 2.1, without loss of generality we may assume that there exists a solution $\{x_n\}$ of equation (1) such that $x_{n-\theta} > 0$ for all $n \geq n_1 \geq n_0$. Let $z_n = x_n + c_n\, x_{n-k}$. Then, in view of condition (c_1), we have $z_n > 0, \Delta(a_n\,(\Delta z_n)^{\alpha}) \leq 0$ for all $n \geq n_1$. Therefore $\{a_n\,(\Delta z_n)^{\alpha}\}$ is a decreasing sequence, and in view of condition (c_2), we have $\Delta z_n > 0$ for $n \geq n_1$. Hence

$$z_n > 0,\ \Delta z_n > 0,\ \Delta(a_n\,(\Delta z_n)^{\alpha}) \leq 0,\ n \geq n_1. \tag{20}$$

Further, we have $x_n \geq (1-c_n)z_n$. Now from equation (1), we obtain

$$\Delta(a_n\,(\Delta z_n)^{\alpha}) + Q_n\, z_{n+1-\ell}^{\beta} \leq 0,\ n \geq n_1. \tag{21}$$

From (21) and (20), we have

$$Q_n\, z_{n+1-\ell}^{\beta} \leq a_n\,(\Delta z_n)^{\alpha} - a_{n+1}\,(\Delta z_{n+1})^{\alpha}$$
$$\leq a_n\,(\Delta z_n)^{\alpha} \leq a_{n-\ell}\,(\Delta z_{n-\ell})^{\alpha}$$

Or $$\left(\frac{Q_n}{a_{n-\ell}}\right)^{\frac{1}{\alpha}} \leq \frac{\Delta z_{n-\ell}}{z_{n+1-\ell}^{\frac{\beta}{\alpha}}}. \tag{22}$$

Since $z_{n-\ell} \leq t \leq z_{n+1-\ell}$, we have

$$\int_{z_{n-\ell}}^{z_{n+1-\ell}} \frac{dt}{t^{\frac{\beta}{\alpha}}} \geq \frac{\Delta z_{n-\ell}}{z_{n+1-\ell}^{\frac{\beta}{\alpha}}}. \tag{23}$$

Using (23) in (22), and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} \left(\frac{Q_s}{a_{s-\ell}}\right)^{\frac{1}{\alpha}} \leq \frac{\alpha}{(\beta-\alpha)}\, z_{n_1-\ell}^{1-\frac{\beta}{\alpha}} < \infty$$

for large values of n. This however contradicts condition (19). This completes the proof of the theorem.

III. Examples

In this section, we present examples to illustrate the main results obtained in the previous section.

Example 1: Consider the difference equation

$$\Delta\left(n\left(\Delta\left(x_n + \frac{1}{2}\, x_{n-2}\right)\right)\right) + \frac{12\,n^2 + 10\,n + 5}{2\,(n-3)^3}\, x_{n-3}^{3} = 0,\ n \geq 4. \tag{24}$$

Here

$$c_n = \frac{1}{2},\ \alpha = 1,\ \beta = 3,\ a_n = n,\ k = 2,\ \ell = 4 \quad and$$
$$q_n = \frac{12\,n^2 + 10\,n + 5}{2\,(n-3)^3}.$$

We see that all conditions of Theorem 2.2 are satisfied, and hence every solution of equation (24) is oscillatory. Infact $\{x_n\} = n\,(-1)^n$, is one such solution of equation (24).

Example 2: Consider the difference equation

$$\Delta\left(\frac{1}{n}\left(\Delta\left(x_n + \frac{1}{3}\, x_{n-2}\right)\right)^3\right) + \frac{512\,(2\,n^2 + 2\,n + 1)}{243\,(n-1)^5}\, x_{n-1}^{5} = 0, n \geq 2. \tag{25}$$

Here

$$c_n = \frac{1}{3},\ \alpha = 3,\ \beta = 5,\ a_n = \frac{1}{n},\ k = 2,\ \ell = 2$$
$$and \quad q_n = \frac{512\,(2n^2 + 2\,n + 1)}{243\,(n-1)^5}.$$

We see that all conditions of Theorem 2.3 are satisfied. Hence every solution of equation (25) is oscillatory and infact $\{x_n\} = 3\,n\,(-1)^n$, is one such solution of equation (25).

IV. Conclusion

In this paper we have established suffcient conditions for the oscillation of all solutions of equation (1) with the condition

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$$

Examples illustrating the results are also presented. The result obtained in Theorem 2.1 improves Theorem 1 of [3] and Theorem 2.1 of [6]. Further it would be interesting to obtain similar results for the equation (1) under the condition $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty$.

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